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ON STATICALLY POSSIBLE FIELDS IN A SIMPLY CONNECTED VOLUME*

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The general continuously differentiable solution of the system of equations

$$\nabla\cdot\mathbf{T}=0\ \text{in}\ V\subset R^3,\quad \mathbf{v}\cdot\mathbf{T}=0\ \text{on}\ \partial V \tag{0.1}$$

is determined, where $\mathbf{v}$ is the normal to ∂V (the boundary ∂V is simply connected). If $\mathbf{T}$ is a vector, then the solution represents the general form of the velocity field of the flow of an incompressible liquid in a closed volume. However, if $\mathbf{T}$ is a symmetric tensor of rank two, then the solution represents the general form of a statically possible field of stresses in a body whose surface is free from any loads (for example, the temperature or residual stresses). The solution can also be used to construct admissible variations of the field of stresses in order to obtain numerical solutions in terms of stresses of boundary-value problems in theory of elasticity. All tensor-vector- and scalar-valued functions considered below are assumed to be smooth, i.e., it is assumed that the functions have continuous derivatives of any order that will be used.

1. Let $\mathbf{u}=\mathbf{u}(\mathbf{x})$ be a field defined on $V_1\supset V$ and let $\omega=\omega(\mathbf{x})$ be a function satisfying the following conditions:

$$\omega(\mathbf{x})>0,\ \mathbf{x}\in\operatorname{int}V;\ \omega(\mathbf{x})=0,\ \partial\omega/\partial\mathbf{v}\neq0,\ \mathbf{x}\in\partial V;\ \omega(\mathbf{x})<0,\ \mathbf{x}\,\overline{\in}\,V\cup\partial V \tag{1.1}$$

It can be shown that there are functions ω that satisfy (1.1) for any domain V with a smooth boundary ∂V. However, from the point of view of the constructiveness of the general solutions given below, it is important to have formulae for evaluating ω. Methods of constructing such formulae for domains of practically any form are discussed in /1/.

Lemma. Let $f=f(\mathbf{x})$ be a bounded function on V_1 such that

$$\nabla^k f=\underbrace{\nabla(\ldots\nabla}_{k\ \text{times}}(\nabla f)\ldots)=\mathbf{0},\quad k=0,1,\ldots,n$$

(where $\nabla^0 f=f$) on the smooth surface ∂V, and let ω satisfy conditions (1.1). Then $f=\omega^{n+1}g$, where g is also bounded on V_1.

The proof of the lemma follows from the generalized Taylor formula /1/. We remark that the lemma also holds the case where f and g are vector- or tensor-valued functions.

Theorem 1. Let ∂V be a smooth and simply connected surface and let ω satisfy (1.1). Then $\mathbf{u}=\mathbf{T}$ is a solution of (0.1) if and only if

$$\mathbf{u}=\nabla\times(\omega\mathbf{p}) \tag{1.2}$$

for some vector field $\mathbf{p}=\mathbf{p}(\mathbf{x})$.

Proof. If $\mathbf{u}$ is defined by (1.2), then (0.1) can be verified by substituting the above expression.

We shall prove the necessity. By virtue of the first equality in (0.1), $\mathbf{u}$ can be represented in the form $\mathbf{u}=\nabla\times\mathbf{q}$ for some $\mathbf{q}=\mathbf{q}(\mathbf{x})$. We find from the second equality in (0.1) that $\mathbf{v}\cdot(\nabla\times\mathbf{q})=0$ on ∂V. Integrating this equality over a surface $S\subset\partial V$ and applying Stokes's theorem, we get

$$0=\iint_S\mathbf{v}\cdot\mathbf{u}\,ds=\iint_S\mathbf{v}\cdot(\nabla\times\mathbf{q})\,ds=\iint_S d\mathbf{s}\cdot(\nabla\times\mathbf{q})=\oint_L d\mathbf{x}\cdot\mathbf{q} \tag{1.3}$$

where the last integral is taken over the contour L, being the boundary of S. Since ∂V is simply connected, it follows that $\oint d\mathbf{x}\cdot\mathbf{q}=0$ for any closed contour on ∂V. Thus, the integral

$$h(M_1)=\int_{MM_1}d\mathbf{x}\cdot\mathbf{q} \tag{1.4}$$

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over a curve MM_1 on ∂V depends only on the position of the initial and final points M and M_1 of MM_1. If M is fixed, (1.4) defines a single-valued function $h(M_1)$ on ∂V.

We extend $h(M_1)$ to a function defined on the entire domain V_1 in such a way that

$$\partial h/\partial \mathbf{v} = \mathbf{v}\cdot\mathbf{q} \quad \text{on} \quad \partial V \tag{1.5}$$

and we set $\mathbf{r}=\mathbf{q}-\nabla h$. It follows from (1.4) and (1.5) that $\nabla h=\mathbf{q}$ on ∂V. Thus, $\mathbf{r}=0$ on ∂V and by the lemma, $\mathbf{r}=\omega\mathbf{p}$. Moreover, $\nabla\times(\omega\mathbf{p})=\nabla\times\mathbf{r}=\nabla\times(\mathbf{q}-\nabla h)=\nabla\times\mathbf{q}=\mathbf{u}$, as required.

Formula (1.2) with an arbitrary $\mathbf{p}$ gives the general expression for the velocity of the flow of an incompessible liquid in a volume V bounded by impermeable walls ∂V under the assumption that ∂V is smooth and simply connected. It can be seen from the proof that in the case of a multiply connected boundary ∂V, the velocity fields that can be obtained in the form (1.2) are those and only those that satisfy condition (1.3), i.e., such that the flux of the liquid through any closed contour on ∂V is equal to zero.

Suppose that ∂V is a doubly connected surface, L is a closed curve on ∂V incontractible to a point on ∂V, $S\subset V$ is a surface spanned by L, and a particular solution $\mathbf{u}_0$ of (0.1) is known such that $\iint\limits_S d\mathbf{s}\cdot\mathbf{u}_0\neq 0$. In this case the general solution of (0.1) can be obtained in the form $C\mathbf{u}_0+\mathbf{u}$, where C is any constant and $\mathbf{u}$ is defined by (1.2).

For example, the velocity of the flow of an incompressible liquid in a closed torus $V=\{\mathbf{x}\mid\omega(\mathbf{x})\geqslant 0\}$, where $\omega(\mathbf{x})=1-x_3^2-(\sqrt{x_1^2+x_2^2}-2)^2$, can be obtained in the form $\mathbf{u}=C\mathbf{u}_0+\nabla\times(\omega\mathbf{p})$, where $\mathbf{u}_0=(-x_2,x_1,0)$, C is an arbitrary constant, and $\mathbf{p}(\mathbf{x})$ is an arbitrary vector-valued function.

The formulation and the proof of Theorem 1 remain the same in the case where $\mathbf{u}$ and $\mathbf{p}$ are tensors of rank two.

2. Let $\mathbf{T}=\mathbf{T}^*=\mathbf{T}(\mathbf{x})$ be a symmetric tensor of rank two bounded for $\mathbf{x}\in V_1$. We shall consider the problem of finding the general solution of the system of Eqs.(0.1).

The symmetric tensor $\mathbf{T}$ can be interpreted as the stress tensor in a continuous medium. Then the first equation in (0.1) is the differential equation of the state of equilibrium of the medium, and the second equation in (0.1) means that there are no surface loads.

Theorem 2. Let ∂V be a smooth and simply connected surface, and let ω satisfy condition (1.1). Then $\mathbf{T}$ is a solution of (0.1) if and only if

$$\mathbf{T}=\mathrm{Ink}(\omega^2\mathbf{R})=\nabla\times(\nabla\times(\omega^2\mathbf{R}))^* \tag{2.1}$$

for some symmetric tensor field $\mathbf{R}$.

Proof. A symmetric tensor field $\mathbf{T}$ is a solution of the first equation in (0.1) if and only if

$$\mathbf{T}=\mathrm{Ink}\,\mathbf{Q} \tag{2.2}$$

for some symmetric tensor $\mathbf{Q}$ /2/. Thus, if $\mathbf{T}$ is defined by (2.1), then the first equality in (0.1) holds. Moreover, evaluating (2.1) with the use of conditions (1.1), we get

$$\mathbf{v}\cdot\mathbf{T}=\mathbf{v}\cdot(-2\nabla\omega\times(\mathbf{R}^*\times\nabla\omega))=0 \quad \text{on} \quad \partial V$$

since $\mathbf{v}\parallel\nabla\omega$.

Therefore, the sufficiency is proved.

We shall prove the necessity. Let $\mathbf{T}$ satisfy equations (0.1) and let the tensor $\mathbf{Q}$ in (2.2) be defined. Applying Stokes's formula, we find from (2.2) and from the second equation in (0.1) that

$$\oint\limits_L d\mathbf{x}\cdot(\nabla\times\mathbf{Q})^*=\iint\limits_S d\mathbf{s}\cdot(\nabla\times(\nabla\times\mathbf{Q})^*)=\iint\limits_S ds\mathbf{v}\cdot\mathbf{T}=0 \tag{2.3}$$

$$\oint\limits_L d\mathbf{x}\cdot(\mathbf{Q}-(\mathbf{x}_1-\mathbf{x})\times(\nabla\times\mathbf{Q}))^*=\iint\limits_S d\mathbf{s}\cdot(\nabla\times(\mathbf{Q}-(\mathbf{x}_1-\mathbf{x})\times(\nabla\times\mathbf{Q}))^*)=$$

$$\iint\limits_S d\mathbf{s}\cdot(\nabla\times\mathbf{Q}^*+(\nabla\times(\nabla\times\mathbf{Q})^*)\times(\mathbf{x}_1-\mathbf{x})-\nabla\times\mathbf{Q})=$$

$$\iint\limits_S ds\mathbf{v}\cdot(\mathbf{T}\times(\mathbf{x}_1-\mathbf{x}))=\iint\limits_S ds\,(\mathbf{v}\cdot\mathbf{T})\times(\mathbf{x}_1-\mathbf{x})=0$$

where L is a closed contour, being the boundary of a surface $S\subset\partial V$, $\mathbf{x}_1$ is fixed, and $\mathbf{x}$ is the current point (with respect to which integration is carried out). By virtue of (2.3), since ∂V is simply connected, the integrals

$$\mathbf{p}(\mathbf{x}_1)=\int\limits_{MM_1} d\mathbf{x}\cdot(\nabla\times\mathbf{Q})^* \tag{2.4}$$

$$\mathbf{q}(\mathbf{x}_1)=\int\limits_{MM_1} d\mathbf{x}\cdot(\mathbf{Q}-(\mathbf{x}_1-\mathbf{x})\times(\nabla\times\mathbf{Q}))^*$$

over a curve $MM_1 \subset \partial V$, where $M_1 = M_1(\mathbf{x}_1)$ and M is a fixed point, define single-valued vector functions $\mathbf{p}(\mathbf{x}_1)$ and $\mathbf{q}(\mathbf{x}_1)$ on ∂V. If $\mathbf{p}$ and $\mathbf{q}$ were defined by (2.4) everywhere on V_1, then the derivative of $\mathbf{q}$ in any direction I could be found by evaluating the derivative of the second integral in (2.4):

$$\frac{\partial \mathbf{q}}{\partial \mathbf{l}} = \mathbf{l}\cdot\mathbf{Q}^* + \Big(\int\limits_{MM_1} d\mathbf{x}\cdot(\nabla\times\mathbf{Q})^*\Big)\times\mathbf{l} = \mathbf{l}\cdot\mathbf{Q} + \mathbf{p}\times\mathbf{l} = \mathbf{l})\cdot(\mathbf{Q} - \mathbf{E}\times\mathbf{p})$$

where $\mathbf{E}$ is the unit tensor, and the equalities

$$\nabla\mathbf{q} = \mathbf{Q} - \mathbf{E}\times\mathbf{p}, \quad \nabla(\nabla\mathbf{q}) = \nabla(\mathbf{Q} - \mathbf{E}\times\mathbf{p}) \tag{2.5}$$

would hold.

But since formulae (2.4) define $\mathbf{p}$ and $\mathbf{q}$ on ∂V only, equalities (2.5) are, generally speaking, not valid for an arbitrary extension of $\mathbf{p}$ and $\mathbf{q}$.

We extend $\mathbf{p}$ and $\mathbf{q}$ onto V_1 in such a way that

$$\partial\mathbf{p}(\mathbf{x}_1)/\partial\mathbf{v} = \mathbf{v}\cdot(\nabla\times\mathbf{Q})^*, \quad \partial\mathbf{q}(\mathbf{x}_1)/\partial\mathbf{v} = \mathbf{v}\cdot(\mathbf{Q} - \mathbf{E}\times\mathbf{p}) \tag{2.6}$$

$$\partial^2\mathbf{q}(\mathbf{x}_1)/\partial\mathbf{v}^2 = \mathbf{v}\cdot(\mathbf{v}\cdot(\nabla(\mathbf{Q} - \mathbf{E}\times\mathbf{p})))$$

for $\mathbf{x}_1 \in \partial V$. Since the derivatives of $\mathbf{p}(\mathbf{x}_1)$ and $\mathbf{q}(\mathbf{x}_1)$ in directions orthogonal to $\mathbf{v}$ are completely defined by the integrals in (2.4) for $\mathbf{x}_1 \in \partial V$ and derivatives (2.6) are consistent with (2.5), it follows that for such an extension the differential formulae (2.5) hold on ∂V.

Now, we consider the symmetric tensor $\operatorname{def}\mathbf{q} = {}^1/_2(\nabla\mathbf{q} + (\nabla\mathbf{q})^*)$ and we set $\mathbf{N} = \mathbf{Q} - \operatorname{def}\mathbf{q}$. By virtue of (2.5), we have

$$\mathbf{N} = \mathbf{Q} - {}^1/_2(\nabla\mathbf{q} + (\nabla\mathbf{q})^*) = \mathbf{Q} - {}^1/_2(\mathbf{Q} - \mathbf{E}\times\mathbf{p} + \mathbf{Q}^* - (\mathbf{E}\times\mathbf{p})^*) = 0$$

$$\nabla\mathbf{N} = \nabla\mathbf{Q} - {}^1/_2(\nabla(\mathbf{Q} - \mathbf{E}\times\mathbf{p}) + \nabla(\mathbf{Q} - \mathbf{E}\times\mathbf{p})^*) = 0$$

on ∂V. Therefore, the lemma yields $\mathbf{N} = \omega^2\mathbf{R}$. Moreover,

$$\operatorname{Ink}(\omega^2\mathbf{R}) = \operatorname{Ink}(\mathbf{Q} - \operatorname{def}\mathbf{q}) = \operatorname{Ink}\mathbf{Q} = \mathbf{T}$$

as required.

With the aid of formula (2.1), one can construct admissible variations of the field of stresses in order to solve boundary-value problems in the mechanics of a deformable rigid body on the basis of the Castilliagno principle. However, there are difficulties arising in the applications of the formula connected with some restrictions on the smoothness and connectivity of the boundary ∂V and with the redundancy of representation (2.1): the tensor $\mathbf{R}$ on the right-hand side of (2.1) contains six arbitrary components, while the six components of $\mathbf{T}$ are connected by three equations (the first equality in (0.1)).

In order to solve plane and axially-symmetric problems of the theory of elasticity, a package of computer programs using variations of the form (2.1) have been designed. For these types of problems, the difficulties have been overcome successfully. Numerical experiments that have been carried out indicate that the approximate solutions of the problems in terms of stresses obtained on the basis of (2.1) converge faster than the solutions of the problems in terms of displacements (which are also based on function (1.1) and on the R-functions).

If Q is regarded as the tensor of small deformations, then the second equation in (0.1) and Eq.(2.2) serve as compatibility conditions for deformations on the surface ∂V. In this case the vector field q defined by (2.4) and (2.6) is the field of displacements, which is connected with the deformations Q on ∂V by the Cauchy equations.

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